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OPTIMAL STABILIZATION POLICIES FOR LESS DEVELOPED ECONOMIES WITH RATIONAL EXPECTATIONS

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This paper investigates the optimal deflationary strategy for a 'financially repressed', less developed economy. In accordance with recent theoratical work, it is assumed that inflationary expectations are rational, although the assumption of adaptive expectations would yield qualitatively similar results. The optimizing time-paths of the two policy instruments, the rate of monetary expansion and the average nominal interest rate paid on money holdings, are explicitly derived and qualitatively characterized. We demonstrate that suitable manipulation of these instruments not only generates a smooth decline of the inflation rate to its target value, but also ensures that real output transitionally either increases or temporarily is constant, but never declines.

The difficulties encountered in efforts to stabilize inflation-prone less developed economies are well-known to students of such economies. If one accepts the monetarist contention that persistently high rates of inflation are ultimately attributable to high rates of monetary expansion, it ineluctably follows that reductions in the latter are a key ingredient in the process of price stabilization. Unfortunately, appreciable reductions in the rate of new-money creation have, in the short run, generated a severe recessionary impact upon these economies, which they can ill-afford.

The analytical issues involved in this dilemma have been examined in a recent paper by the present author (1976a; see also his 1976b) in which, following upon the work of R.I. McKinnon (1973), central emphasis was placed on fluctuations induced by monetary policy in the flow of commercial bank loans for the financing of net as well as replacement investment in working capital by productive enterprises. It was demonstrated in Kapur (1976a) that a discrete reduction in the rate of monetary expansion would indeed precipitate a short-run reduction in the level of real output: however, an alternative policy – a discrete increase in the average nominal interest rate paid on money-holdings – would not only immediately lower the rate of inflation and increase the level of real output, but would also, through its favourable effect upon bank intermediation and thence upon the steady-state rate of growth of real output, lower the steady-state rate of inflation in the economy.

While suggestive, the analysis in Kapur (1976a) suffers from what I now believe to be two basic weaknesses. In the first place, it was predicated upon the assumption that inflationary expectations evolve adaptively. The adaptive expectations formulation has come under severe criticism in recent years, on the grounds that it implies a rather machanistic view of the manner in which economic agents form their expectations. Secondly, the previous analysis concentrated solely upon a comparison of the macro-dynamic effects of a once-and-for-all reduction in the monetary growth rate with those of a once-and-for-all increase in the interest rate on money-holdings. In a realistic stabilization program, however, one would certainly expect both of these instruments to be dynamically manipulated so as to achieve the desired pattern of evolution of the entire economic system towards its new steady-state equilibrium.

The present paper addresses itself to the task of remedying both of these short-comings. In Section I, we jettison the adaptive expectations assumption, and re-formulate the structural model on the basis of the assumption that expectations are rational, which, in the present deterministic model, is equivalent to assuming perfect foresight. While this may appear to be an unduly extreme idealization, it does provide an approximation to the direction towards which expectations would tend as economic agents become better-informed: alternatively, it would hold exactly if the government were to announce beforehand the planned evolution of the inflation rate (and if such announcements were held to be credible). In Section II, we formulate the optimal control problem of achieving the most preferred pattern of transition of the economy to a lower inflation rate (and a higher growth rate), in terms of minimizing a specified loss function, and we derive the necessary and sufficient conditions for optimality. In Section III, an examination of the necessary conditions provides the basis for a synthesis of the optimal transition path of the economy, while Section IV, finally, presents some brief concluding observations.

I

Let us proceed first to the specification of the formal model, which overlaps considerably with that presented in Kapur (1976a). Aggregate real output Y of the economy is assumed to be related to the sum of the flow of working capital and the services of fixed capital, denoted by K, by means of a simple Harrod-Domar production function:

$$Y = \sigma K \tag{1}$$

where σ is the output-capital ratio. We assume that the fixed capital stock is under-utilized and that the services of fixed capital constitute a technologically fixed fraction α of the total flow K: hence the availability of working capital constitutes the operative constraint on the level of output.[1]

Next, it is convenient to express the nominal supply of money M in terms of its determinants – the total availability of high-powered money in the system, and the credit expansion process. Thus:

$$M = C + L \tag{2}$$

where C comprises the government's fiduciary issue while L is equal to the total outstanding volume of loans made by the commercial banking system. For simplicity, we shall assume that the public's desired and actual currency-deposit ratio and the banks' desired and actual reserve-deposit ratio are unchanged through time, so that L/M and C/M are constant: let L/M = q, so that C/M = 1 – q. C is assumed to be issued as transfer payments, and its rate of issue $\dot{C}/C$, which we denote as μ, is a policy parameter: the constancy of q then implies that $\dot{L}/L = \dot{M}/M = \mu$ as well.

The rate of increase of the total (flow) usage of capital is given by:

$$\begin{aligned} \dot{K} &= \frac{1}{1-\alpha}\left[\frac{\dot{L} - \dot{P}\,\theta(1-\alpha)K}{P}\right] \\ &= \frac{1}{1-\alpha}\left[\mu\, q\, \frac{M}{P} - \pi\,\theta(1-\alpha)K\right], \end{aligned} \tag{3}$$

where P is the price-level, π is the rate of inflation $\dot{P}/P$ (dots denote time derivatives), and where θ is a positive parameter representing the fraction of the inflation-induced increase in the cost of replacing 'used up' working capital $(1-\alpha)K$ at any time that is financed through borrowing from banks (the remainder of the cost increase being assumed to be financed through the internal resources of productive enterprises). Equation (3) is predicated upon

[1] 'Working capital' in turn comprises inputs of raw materials and goods in process as well as advances to workers prior to actual sales. Relative input prices are assumed to be fixed (for example, labour is assumed to be in excess supply at a fixed real wage), so that aggregation into a composite working capital input is permissible.

the assumption that net investment in working capital is financed solely by commercial bank lending, which also finances the fraction θ of the increased cost of replacement investment: the square-bracketed expression thus represents the real amount of net investment in woking capital, obtained by deducting from the flow of new loans $\dot{L}$ that portion of it that is used to defray the increased cost of bank-fianced replacement investment. Since fixed capacity is in excess supply, $\dot{K}$ is then simply equal to $1/(1-\alpha)$ times net working capital investment.

The desired level of real cash balances $(\frac{\hat{M}}{P})$ is assumed to be governed by a Cagan-type (1956) function:

$$(\frac{\hat{M}}{P}) = Y\ e^{-a\ (\pi^* - d)} \tag{4}$$

where a is a positive parameter, π^* is the expected rate of inflation and d is the average nominal deposit rate on money-holdings. Turning next to the rate of growth of real output, $\dot{Y}/Y$, which we denote by γ, this may be obtained by dividing equation (3) by K, since from equation (1) $\dot{Y}/Y = \dot{K}/K$:

$$\gamma = \mu \frac{\sigma q}{1-\alpha} \frac{M}{PY} - \pi\, \theta \tag{5}$$

Logarithmically differentiating the identity MV=PY with respect to time, we have:

$$\mu + \frac{\dot{V}}{V} = \pi + \gamma \tag{6}$$

In steady-state equilibrium, V is constant, so that $\pi = \mu - \gamma$: moreover, actual and desired money-holdings are then equal as also are expected and actual rates of inflation. From equation (4) and (5), we thus have, *in the steady-state:*

$$\gamma = \mu \frac{\sigma q}{1-\alpha} e^{-a\ (\mu - \gamma - d)} - \mu\, \theta + \gamma\, \theta \tag{7}$$

Equations (1) – (7) are common both to the model developed in this paper, and that presented in Kapur (1976a). The latter was completed by the incorporation of the well-known adaptive expectations assumption:

$$\frac{d\pi^*}{dt} = \beta(\pi - \pi^*) \tag{8}$$

and by a variant of the accelerationist hypothesis [see Goldman (1972)], whereby π is a function of π^* and of the excess demand for output (which is equal, by Walras' Law, to the excess supply of money, these constituting the only markets in the model):

$$\pi = h \left(\frac{M}{PY} - \frac{\hat{M}}{PY}\right) + \pi^*, \quad h > 0 \tag{9}$$

It is then demonstrated in Kapur (1976a) that the foregoing dynamic model is reducible to self-contained dynamic system in the variables $\log_e V$ (denoted as W) and π^*. We shall discuss the dynamic properties of this system below. Here, however, it is of interest to dwell briefly on the contrasting implications of the short-run and the steady-state formulations of γ, as given in equations (5) and (7) respectively. Suppose at time $t=t_o$ there occurs an instantaneous upward shift in μ: since, from equation (9), π does not instantaneously change in response, we have

$$\left.\frac{d\gamma}{d\mu}\right|_{t=t_o} = \frac{\sigma q}{1-\alpha} \frac{M}{PY} > 0 \tag{10}$$

Thus, the growth rate instantaneously increases, essentially on account of the increased flow of real bank loans relative to income $\dot{L}/PY$ (which equals $\mu\, q \frac{M}{PY}$). However, in the steady state, we have, by implicitly differentiating equation (7) with respect to μ:

$$\left.\frac{d\gamma}{d\mu}\right|_{\substack{\text{Across}\\ \text{steady}\\ \text{states}}} = \frac{\frac{\sigma q}{1-\alpha} e^{-a(\mu-\gamma-d)} [1 - a\mu] - \theta}{1 - a\mu \frac{\sigma q}{1-\alpha} e^{-a(\mu-\gamma-d)} - \theta} \tag{11}$$

This rather complex equation is analyzed in detail in Kapur (1976b; pp.206-207), where it is shown that $d\gamma/d\mu$ is positive for low values of μ, and negative for high values of μ. Thus, in the latter region (which primarily concerns us), an increase in μ, while it raises γ in the short run, *lowers* it in the steady state, on account firstly of the induced increase in π [which enters negatively in equation (5)], and secondly of the resultant shrinkage in $\frac{M}{PY}$, which reduces the real flow of loans $\mu\, q\frac{M}{PY}$. This contrast between short- and long-run results also carries over to the present model, to which we now turn.

As pointed out in the Introduction, we propose in this paper to replace the adaptive expectations formulation of equation (8) with that of perfect foresight:

$$\pi^* = \pi \tag{12}$$

If we combine equation (12) with equation (9), we encounter certain difficulties, inasmuch as the conjunction of these two equations implies that equilibrium is continuously maintained between desired and actual real money balances. This can be shown to generate anomalous results (for example, it implies that a discrete incease in d will produce an instantaneous equal increase in the expected as well as the actual rate of inflation, which is quite inexplicable from an economic stand-point). Accrodingly, we replace equation (9) with the following closely analogous formulation:

$$\frac{d\pi}{dt} = \delta\,(W^* - W), \qquad \delta > 0, \tag{13}$$

where W^* is the logarithm of the desired level of velocity, and hence, from equations (4) and (12), is equal to $a(\pi - d)$ while $W = \log_e V$. Equation (13) is based upon the view that sellers, as a consequence of prolonged exposure to an inflationary economic system, have adopted the practice of raising their prices in each time-period simply as a matter of normal procedure: the rate of price increase is, however, varied in response to the level of excess demand for goods or supply of money (relative to the level of total output), which is directly correlated with $(W^* - W)$.[2]

Our revised structural model is thus defined by equations (1) – (7) and (12) – (13). By replacing M/PY by $\bar{e}^{W}$ in equation (5), and $\dot{V}/V$ by $\dot{W}$ in equation (6), we obtain, after substituting from equation (5) into equation (6) and re-arranging terms:

$$W = (1 - \theta)\pi - \mu\,[1 - \frac{\sigma q}{1-\alpha}\,\bar{e}^{W}] \tag{14}$$

[2]Our formulation is inspired by equation (4) of Obst (1978). Obst, however, assumes that $d\pi/dt$ depends upon the difference between actual and desired nominal holdings of money, relative to the nominal supply of money. It would appear to be economically more reasonable, however, to represent the 'demand pressure' for goods by the difference between actual and desired money holdings relative to the level of total output. This is the view adopted in equation (13) above except that, for reasons of mathematical convenience, demand pressure is represented as a linear function of W instead of M/PY.

Similarly, replacing W^* by $a(\pi - d)$ in equation (13) we have:

$$\frac{d\pi}{dt} = \delta\,[a(\pi - d) - W] \qquad (15)$$

Equations (14) and (15) now constitute a self-contained dynamic system in the two variables W and π. Assuming that a steady-state solution to these two equations exists and is differentiable, they can be easily shown to imply the following phase-diagram:

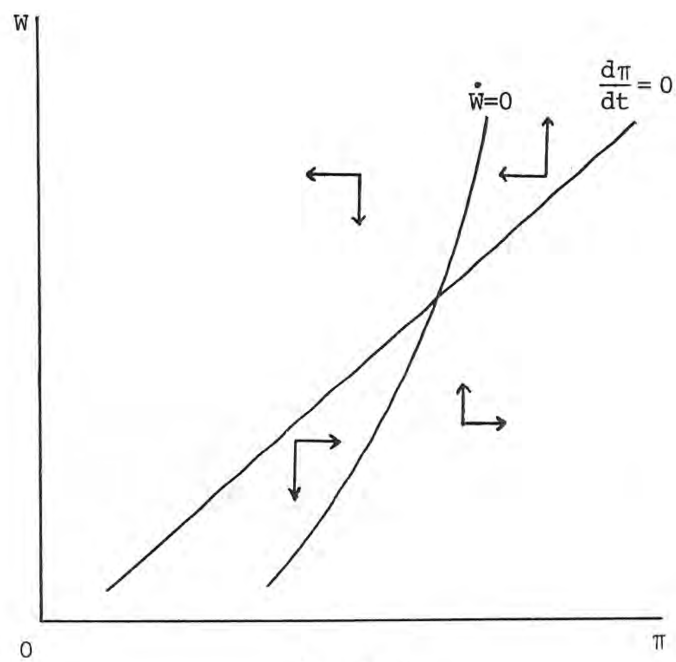

Figure 1

The dynamic effects of shifts in the policy parameters μ and d may thus be ascertained through suitable manipulation of the loci in Figure 1. It turns out that the results in the present model are perfectly clear-cut, unlike the situation with regards to the model discussed in Kapur (1976a) where recourse had to be made to numerical simulations in order to resolve certain ambiguities in the analytical treatment. This difference is attributable to the fact that the above phase-diagram is in W-π space, whereas those in Kapur

(1976a) [which correspond to equations (10) – (11) above] were in $W - \pi^*$ space: further investigation was thus called for in the latter to determine the effects of policy upon π and γ.

Thus, in the present model it is easily demonstrated that a discrete reduction in μ produces a leftward shift of the $\dot{W}=0$ locus, while leaving the $d\pi/dt=0$ locus unaffected. This generates an initial phase of rising W and falling π, followed by one of falling W and π, and then a possibly cyclical covergence to the new steady-state. Moreover, from equation (5) (with M/PY replaced by e^{-W}) it is seen that the reduction in μ precipitates an instantaneous fall in γ, which may either rise or fall further in the first phase of rising W and falling π, while it definitely rises in the subsequent phase of falling W and π. Next, an increase in d leads to a downward shift of the $d\pi/dt=0$ locus, with no shift in the $\dot{W}=0$ locus. This causes both W and π to commence falling, and then to converge in a possibly cyclical manner to the new equilibrium. These movements cause γ to rise smoothly from time 0 onwards.

It is interesting to note that these results are qualitatively almost identical to those that simulation analysis has indicated to be characteristic of the model in Kapur (1976a): the only difference of any significance appears to be that, in the present model, in contrast to the previous one, an increase in d does not produce an instantaneous downward jump in π and upward jump in γ, but instead a smooth pattern of evolution of these two variables. Both models concur completely, however, as to the superiority of an increase in d over a reduction in μ as a stabilization device, in that the former generates a more favourable transitional time-path for γ. The 'robustness' of this result in the face of alternative specifications as to the determination of inflationary expectations should considerably enhance our confidence in its general applicability.

II

We proceed next to inquire into the determination of the optimal stabilization path of the economy. As the preceding discussion suggests, the 'state variables' of our optimal control problem are W and π, the motions of which are governed by differential equations (14) and (15) respectively. The control variables are of course d and μ, and it is illuminating to specify firstly the constraints on their use. It is clear that the nominal interest rate paid on money-holdings cannot fall below zero (assuming that currency is costless to hold), which therefore constitutes the lower limit on d. Next, it is shown in Kapur (1976a, p. 789) that assuming that commercial banks are in competitive equilibrium with zero excess profits, that the government pays no interest on bank reserves or on currency in circulation, and that the costs of the banking system are a fixed fraction z of the real supply of money, d must equal q.l − z, where l is the nominal loan rate, since under the stated conditions payment of interest on money balances can only be financed out of interest receipts from bank lending. Since l cannot exceed the expected nominal rate of profit from holdings of working capital, which is $\bar{r} + \pi$ (since $\pi^* = \pi$), $\bar{r}$ being the fixed real rate of profit,[3] it follows that d cannot exceed $q(\bar{r} + \pi) - z$, the latter therefore constituting the (variable) upper bound.

As for the control variable μ, we shall assume that it is subject to a fixed (but fairly high) upper limit $\bar{\mu}$. Since μ refers to the rate of expansion of the nominal supply of money, and not merely of currency alone, this assumption is perhaps not excessively arbitrary. Specification of the lower limit on μ, however, poses a somewhat more delicate problem. As the discussion below indicates, it was discovered to be highly convenient to express the intertemporal objective function in the form of a preference ordering defined over movements in the state variables W and π. This creates certain complications with regard to the time-path of γ which from equation (5) is seen to be a function of μ, W, and π: in fact, it is entirely possible that an ostensibly 'optimal' time-path of W and π might involve a period of time during which γ is negative. To obviate this unappealing contingency, we shall stipulate that (since from equation (5) γ is instantaneously an increasing function of μ), at every point on the optimal path μ should be sufficiently high as to ensure that γ is always non-negative. Setting $\gamma = 0$ in equation (5) and re-arranging terms, the resulting (variable) lower bound on μ is simply $\theta\pi / \frac{\sigma q}{1-\alpha} e^{-W}$.[4]

We proceed lastly to specify the objective function of the policy-maker. Suppose that he were to arbitrarily select as his ultimate targets particular values γ_1 and π_1 of the growth rate and inflation rate respectively, subject only to the conditions that these be steady-state values and that their attainment does not entail the violation of the feasibility constraints on μ and d. Suppose also that the corresponding required value d_1 of d were strictly less than its upper bound, which would be equal to $q(\bar{r} + \pi_1) - z$. If then d were raised 'slightly', while μ were held constant, it can be readily

[3] The derivation of $\bar{r}$ in terms of σ and α is provided in Kapur (1976a, pp. 792-793).

[4] It is shown in Section III below that along the optimal trajectory both W and π are bounded. Since the above lower limit on μ is a continuous function of W and π, it is also bounded. It is therefore possible to choose a value of the upper limit $\bar{\mu}$ that strictly exceeds the value of the lower limit at every point of the optimal trajectory.

shown that the resulting steady-state value of γ would exceed γ_1, while that of π would fall below π_1 (the purpose of making the increase in d sufficiently small is to ensure by continuity that the resulting decrease in π will not be so considerable as to result in the violation of the upper bound on d). Since both increases in γ and reductions in π are presumably desirable on welfare grounds, the originally chosen targets (γ_1, π_1) could not have been 'optimal' in the deeper sense of the term. These considerations suggest that it is reasonable to stipulate that in the final steady-state equilibrium d must exactly equal its corresponding upper bound.

Since in the steady state $\pi = \mu - \gamma$ (as may be readily verified from equation (6) above by noting that in the steady state $\dot{V}/V = 0$), we therefore require that d at that point be equal to $q(\bar{r} + \mu - \gamma) - z$. As d cannot thus be independently manipulated in the steady state, it is not possible to arbitrarily select target values of π and γ, as we have only one remaining instrument μ to achieve these. This, however, does not constitute as severe a restriction as might initially appear to be the case, since from the preceding discussion it is patent that the resulting locus of feasible choices of π and γ constitutes a 'maximal frontier' of these variables, with d being 'optimally' adjusted along each point on the frontier. It therefore behoves the policy-maker to choose an objective on this feasible locus, based upon his judgment as to the desired trade-off between higher growth and inflation rates (it is easily shown that, under certain relatively weak restrictions, the feasible locus consists of an initial segment of rising γ and rising π, followed by a segment of falling γ and rising π: presumably the chosen objective would consist of a point on the former segment). We therefore denote the chosen target by the pair (π^T, γ^T) of desired steady-state values of π and γ. Corresponding to these ultimate desiderata are the target steady-state values W^T [equal to $[a(1-q)\pi^T + a(z-q\bar{r})]$], μ^T (equal to $\pi^T + \gamma^T$), and d^T [equal to $q(\bar{r} + \pi^T) - z$] of the remaining state variable and the control variables respectively, where it is assumed that the targets π^T and γ^T are such that this value of μ lies in the interior of its feasible space.

There remains the problem of specifying the loss function associated with transitional deviations of the above-mentioned variables from their desired values. The most convenient formulation for this purpose is one that is quadratic in the state variables: we therefore postulate that the policy-maker wishes to maximize

$$\int_0^\infty e^{-\rho t}[-m_1 (W - W^T)^2 - m_2 (\pi - \pi^T)^2]\, dt$$

where m_1 and m_2 are positive constants, ρ is the positive social discount rate, and t is an index of time. This formulation might be objected to on the grounds that negative values of $(W - W^T)$ are less undesirable than positive ones, since the lower the value of W, the higher the degree of monetization of the economy (as measured by the money-GNP ratio), and hence, *ceteris paribus*, the higher its instantaneous growth rate. In this paper, however, we assume that the economy is initially in a state of 'financial repression' with W significantly above W^T: since one might therefore surmise that W for the most part will be declining towards W^T (so that positive deviations are experienced most of the time), the approximation entailed in ignoring this refinement does not appear to be excessively misleading from an analytical stand-point.

In summary form, therefore, our complete optimal control problem may be stated as follows:

$$\text{Maximize} \qquad \int_0^\infty e^{-\rho t} \left[-m_1 (W - W^T)^2 - m_2 (\pi - \pi^T)^2 \right] dt$$

$$\text{subject to} \qquad \dot{W} = (1-\theta)\pi - \mu \left[1 - \frac{\sigma q}{1-\alpha} e^{-W}\right]$$

$$\frac{d\pi}{dt} = \delta \left[a(\pi - d) - W\right]$$

$$0 \leq d \leq q(\bar{r} + \pi) - z,$$

$$\theta\pi / \frac{\sigma q}{1-\alpha} e^{-W} \leq \mu \leq \bar{\mu},$$

and to given initial values $W(0)$ and $\pi(0)$ of W and π.

The necessary conditions for an optimal solution to this problem may be derived through an application of the Pontryagin maximum principle [cf. Arrow and Kurz (1970), Chapter 2, Proposition 7]. Forming the current-value Lagrangian

$$\begin{aligned} L = & -m_1 (W - W^T)^2 - m_2 (\pi - \pi^T)^2 + \psi_1(t) \left[(1-\theta)\pi - \mu \left(1 - \frac{\sigma q}{1-\alpha} e^{-W}\right)\right] \\ & + \psi_2(t) \left[\delta(a(\pi - d) - W)\right] + \lambda_1(t) \left[q(\bar{r} + \pi) - z - d\right] \\ & + \lambda_2(t)\, d + \lambda_3(t) \left[\bar{\mu} - \mu\right] \\ & + \lambda_4(t) \left[\mu - \theta\pi / \frac{\sigma q}{1-\alpha} e^{-W}\right], \end{aligned} \tag{16}$$

where the $\psi_i(t)$ (i = 1, 2) and $\lambda_j(t)$ (j = 1, 2, 3, 4) are the auxiliary variables and Lagrangian multipliers respectively, it is necessary that at the maximizing values $\mu(t)$ and d(t) [and the resultant values of W(t) and $\pi(t)$], the following conditions hold:

$$\frac{\partial L}{\partial \mu} = 0; \frac{\partial L}{\partial d} = 0; \tag{17}$$

$\lambda_j(t) \geqslant 0$, for all j; $\lambda_j . F_j = 0$, where $F_j \geqslant 0$ is the j-th functional constraint(j = 1, 2, 3, 4),

$$\dot{\psi}_1(t) = \rho\psi_1(t) - \frac{\partial L}{\partial W} \tag{18}$$

$$\dot{\psi}_2(t) = \theta\psi_2(t) - \frac{\partial L}{\partial \pi} \tag{19}$$

In addition, we retain equations (14) and (15) above, with μ and d replaced by $\mu(t)$ and d(t) respectively. The first-order conditions (17) may be stated explicitly as follows:

$$\begin{aligned} -\psi_1 \ (1 - \frac{\sigma q}{1-\alpha} e^{-W}) - \lambda_3 + \lambda_4 &= 0 \\ -\psi_2 \ a \ \delta - \lambda_1 + \lambda_2 &= 0 \end{aligned} \tag{17a}$$

where the functional dependence of the ψ_i and λ_j on t has been suppressed for convenience of presentation. In Section III below, we subject this entire set of formal conditions to further scrutiny in order to obtain a precise characterization of the optimal path. [5]

III

The optimal deployment of the policy instruments over time, as well as the associated pattern of evolution of the economic system, may most readily be ascertained through an application of the analytical techniques developed by Ryder (1969), and Hamada (1969). An inspection of conditions (17a) reveals that there are a total of five possible alternative configurations of control and state variables, which are exhibited hereunder:

[5] It is further demonstrated there that these necessary conditions are also sufficient for optimality.

Pattern I

Suppose that $\psi_1(t) > 0$. From the former of conditions (17a) we see that this implies that λ_4 is positive (and $\lambda_3 = 0$),[6] so that $\mu = \theta\pi / \frac{\sigma q}{1-\alpha} e^{-W}$. Next, if we assume in addition that $\psi_2(t) < 0$, it follows from the latter of equations (17a) that λ_1 is positive (and $\lambda_2 = 0$), so that $\hat{d} = q(\bar{r} + \pi) - z$. Substituting the functions $\hat{\mu}$ and $\hat{d}$ into equations (14) and (15) and rearranging terms, the evolution of the state variables in this pattern can then be seen to occur according to the following equations:

$$\dot{W} = \pi \left[1 - \theta e^{W} / \frac{\sigma q}{1-\alpha}\right] \tag{20}$$

$$\frac{d\pi}{dt} = \delta \left[a(1-q)\pi - a(q\bar{r}-z) - W\right] \tag{21}$$

These two equations can be shown easily to imply the following phase diagram (Figure 2).

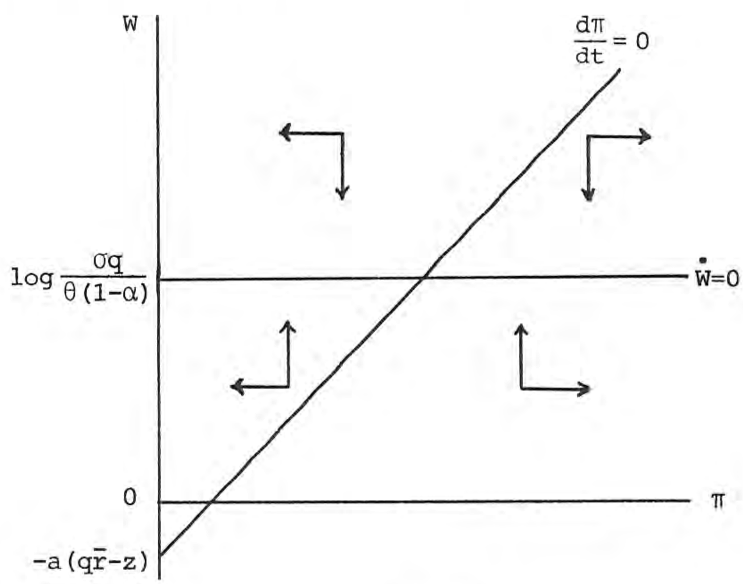

Figure 2

[6] Note that $\sigma q/1-\alpha . e^{-W}$ is simply euqal to the ratio of real bank loans to the real stock of working capital, and hence, from the discussion in Kapur (1976a, f.n. 8), will be strictly less than unity provided that $\pi > 0$. As can be readily seen from the discussion below, we may quite plausibly assume this condition to hold throughout the transition process.

In the above figure, we have assumed, as appears reasonable, that $q\bar{r} - z > 0$, and we have further assumed that $\frac{\sigma q}{\theta(1-\alpha)} > 1$: the latter assumption can be relaxed without affecting any of the qualitative conclusions derived below. We have not depicted the $\pi = 0$ root of the equation $\dot{W} = 0$, since, as mentioned earlier, it is assumed that the parameters of the model are such that π will always be positive.

Lastly, it is necessary to ascertain the time-paths of the auxiliary variables by explicitly evaluating equations (19) above. Since in this pattern λ_2 and λ_3 are equal to zero, it is possible to solve for λ_1 and λ_4 from equations (17a): by partially differentiating equation (16) with respect to W and π respectively, substituting these and the values of $\hat{\mu}$, $\hat{d}$, λ_1 and λ_4, into equations 19), and re-arranging terms, we finally have:

$$\begin{aligned} \dot{\psi}_1(t) &= 2m_1(W - W^T) + \psi_1 \ (\rho + \frac{\theta \pi \, e^W}{\sigma q/(1-\alpha)}) + \psi_2 \, \delta \\ \dot{\psi}_2(t) &= 2m_2(\pi - \pi^T) + \psi_2 \ [\rho - a\delta \ (1\text{-}q)] - \psi_1 [1 - \frac{\theta \, e^W}{\sigma q/(1-\alpha)}] \end{aligned} \tag{19a}$$

Pattern II

Since an analogous procedure is involved in characterizing all the remaining patterns, we can be brief in our presentation of these. In Pattern II we assume that $\psi_1(t)$ is again positive, but that, unlike the preceding case, ψ_2 is also positive. Thus, $\hat{\mu}$ is again equal to the lower bound on μ: however, λ_1 is now equal to zero and λ_2 is positive, so that $\hat{d} = 0$. Equations (14) and (15) thus become:

$$\dot{W} = \pi \, [1 - \theta e^W / \frac{\sigma q}{1-\alpha}] \tag{22}$$

$$\frac{d\pi}{dt} = \delta \ [a\pi - W] \tag{23}$$

The resulting phase diagram is given by Figure 3, while equations (19) become:

$$\begin{aligned} \dot{\psi}_1 &= 2m_1(W - W^T) + \psi_1 [\rho + \frac{\theta \pi \, e^W}{\sigma q/(1-\alpha)}] + \psi_2 \, \delta \\ \dot{\psi}_2 &= 2m_2(\pi - \pi^T) + \psi_2 \ (\rho - a \, \delta) - \psi_1 \ [1 - \frac{\theta \, e^W}{\sigma q/(1-\alpha)} \ ] \end{aligned} \tag{19b}$$

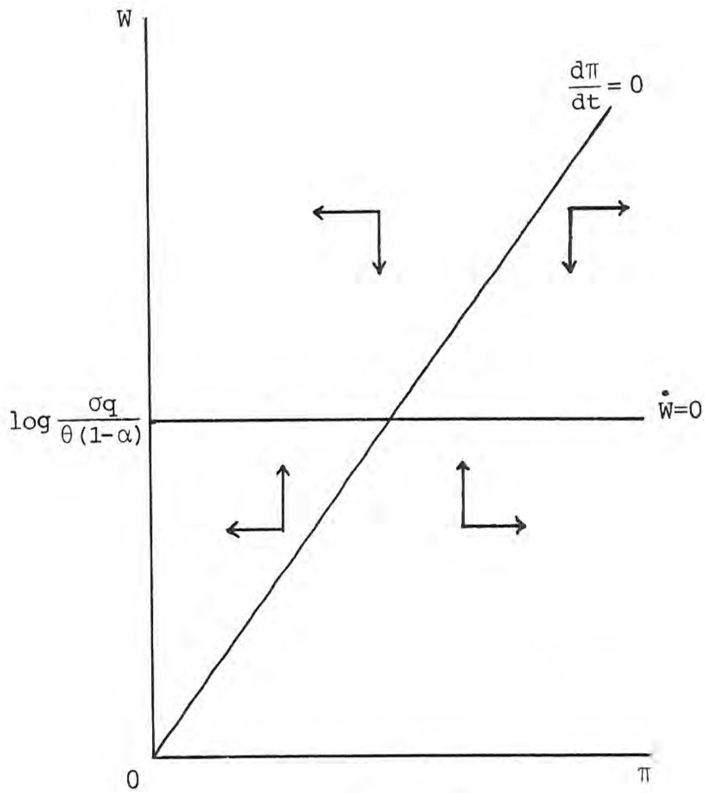

Figure 3

Pattern III

Suppose next that $\psi_1(t)$ is negative. From the former of equations (17a), this implies that λ_3 is now positive and $\lambda_4 = 0$, so that $\hat{\mu} = \bar{\mu}$. Suppose also that $\psi_2(t)$ is positive: from the latter of equations (17a), it then follows that λ_2 is positive and $\lambda_1 = 0$, whereby $\hat{d} = 0$. Substituting these values of $\hat{\mu}$ and $\hat{d}$ in equations (14) – (15) we have:

$$\dot{W} = (1-\theta)\pi - \bar{\mu}\,(1 - \frac{\sigma q}{1-\alpha}\; e^{-W}) \tag{24}$$

$$\frac{d\pi}{dt} = \delta\;[a\pi - W] \tag{25}$$

These imply the following phase-diagram (Figure 4).

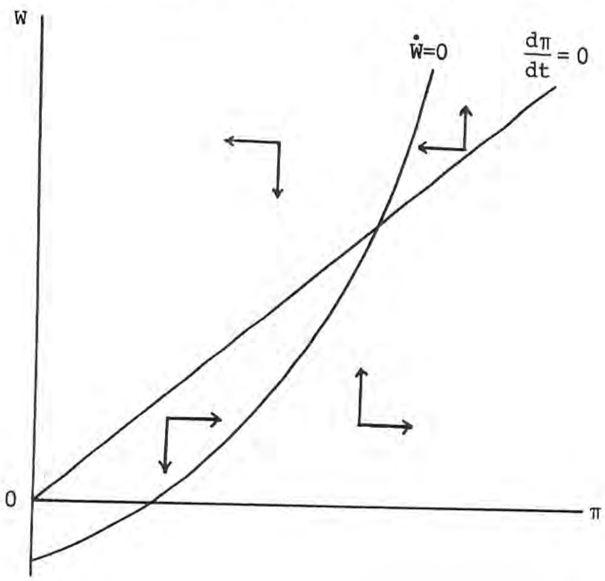

Figure 4

The negative intercept of the $\dot{W} = 0$ locus occurs if we assume that $\frac{\sigma q}{1-\alpha} < 1$: this is not unreasonable given our previous specification that $\frac{\sigma q}{1-\alpha} e^{-W} < 1$ within the experienced range of variation of W. Lastly, equations (19) reduce to:

$$\dot{\psi}_1 = 2m_1 (W - W^T) + \psi_1 [\rho + \bar{\mu} \frac{\sigma q}{1-\alpha} e^{-W}] + \psi_2 \delta$$
$$\dot{\psi}_2 = 2m_2 (\pi - \pi^T) + \psi_2 (\rho - a\delta) - \psi_1 (1-\theta) \qquad (19c)$$

Pattern IV

A further possibility is that $\psi_1(t)$ and $\psi_2(t)$ are both negative. This implies that λ_3 and λ_1 are positive, while $\lambda_4 = 0 = \lambda_2$: thus, $\hat{\mu} = \bar{\mu}$, and $\hat{d} = q(\bar{r}+\pi) - z$. We thus have:

$$\dot{W} = (1-\theta)\pi - \bar{\mu} [1 - \frac{\sigma q}{1-\alpha} e^{-W}] \qquad (26)$$

$$\frac{d\pi}{dt} = \delta [a(1-q)\pi - a(q\bar{r}-z) - W] \qquad (27)$$

and, corresponding phase-diagram is given by Figure 5.

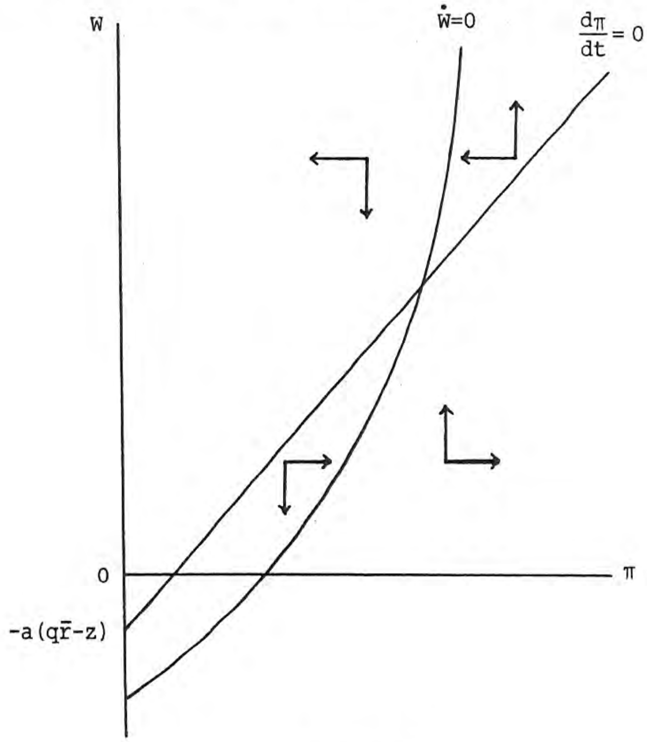

Figure 5

In addition, we have:

$$\begin{aligned}\dot{\psi}_1 &= 2m_1(W - W^T) + \psi_1[\rho + \bar{\mu}\frac{\sigma q}{1-\alpha}\bar{e}^W] + \psi_2\,\delta \\ \dot{\psi}_2 &= 2m_2(\pi - \pi^T) + \psi_2[\rho - a\delta\,(1-q)] - \psi_1(1-\theta)\end{aligned} \tag{19d}$$

Pattern V

Suppose, finally, that $\psi_1(t)$ is equal to zero over an interval of positive duration. From the former of equations (17a) it then follows that $\lambda_3 = \lambda_4$, which is only possible if each is equal to zero. Moreover, over such an interval $\dot{\psi}_1$ must equal zero also: substituting these restrictions into the former of equations (19) and re-arranging terms, we have:

$$\psi_2 = -\frac{2m_1}{\delta}(W - W^T) \tag{28}$$

Suppose next that $W - W^T > 0$: the trajectory corresponding to the alternative case plays no part in the optimal solution, and is hence ignored. From equation (28) ψ_2 will then be negative, which implies from the latter of equations (17a) that $\lambda_1 > 0$ (and $\lambda_2 = 0$), so that $\hat{d} = q(\bar{r}+\pi) - z$. In addition, equation (28) also implies that $\psi_2 = -(2m_1/\delta)\dot{W}$: substituting all these into the latter of equations (19) and manipulating terms we have:

$$\dot{W} = -[a\delta(1-q) - \rho]\ (W - W^T) - \delta \frac{m_2}{m_1} (\pi - \pi^T) \qquad (29)$$

For simplicity, we shall assume that $a\delta(1-q) - \rho > 0$: alternative specifications lead to qualitatively identical results. It might ostensibly appear that equation (29) is incompatible with equation (14), which also purports to describe the motion of W: in fact, however, the two equations may be reconciled through suitable manipulation of the control variable μ, in accordance with the following equation:

$$\mu(1 - \frac{\sigma q}{1-\alpha} e^{-W}) = (1 - \theta)\pi + [a\delta(1-q) - \rho]\ (W - W^T) + \delta \frac{m_2}{m_1} (\pi - \pi^T) \qquad (30)$$

For values of (π, W) that are sufficiently 'close' to the target (π^T, W^T) it is easily shown that the solution for μ from this equation falls within its upper and lower bounds: for 'distant' values of (π, W), however, it is conceivable that the resulting value of μ violates one or the other of these bounds. The implications of this are examined below. Our formal specification of Pattern V is completed with the inclusion of equation (27) above, which continues to apply: in conjunction with equation (29) it implies the following phase digram (Figure 6). Although we have in this figure sketched in the complete phase-plane corresponding to the two equations, it is clear from the foregoing discussion that only that portion of it that lies above the $W = W^T$ axis is of relevance to our optimization problem. From equations (27) and (29) it is readily verified that, as indicated above, the rest-point of this system occurs precisely at (π^T, W^T).[7] Finally, it may be easily confirmed that, within the framework of this dynamical system, (π^T, W^T) constitutes a saddle-point, so that there is a unique trajectory (depicted as AA in Figure 6) that converges to it.

[7] It should be noted that the $d\pi/dt = 0$ locus in Pattern V is identical to those of Patterns I and IV: in each case d is set equal to its (variable) upper bond, which, as indicated earlier, is also a characteristic of the target steady state.

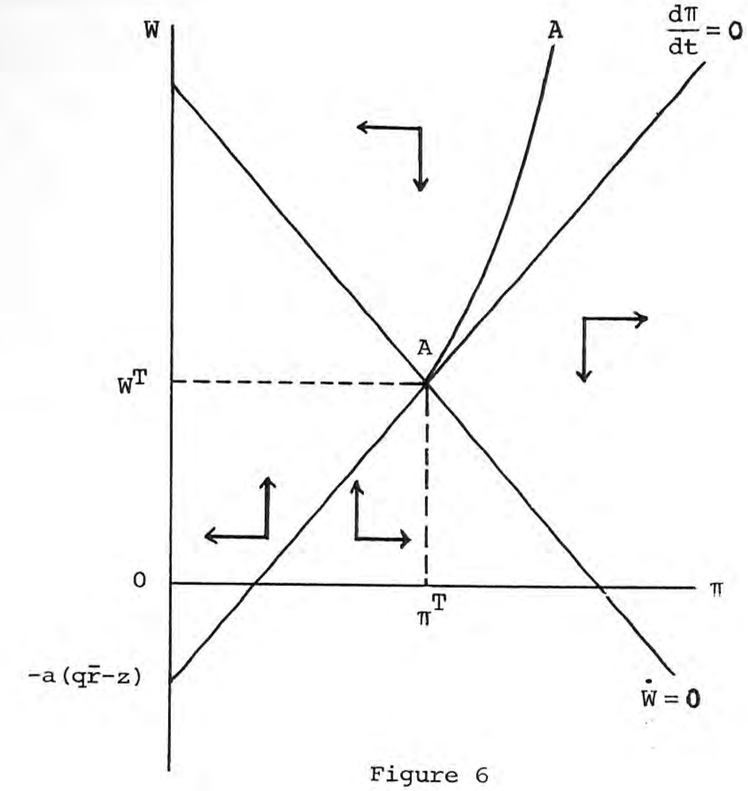

Figure 6

Having delineated these five configurations,[8] we proceed now to the synthesis of the optimal transition path. Clearly, if the initial position of the economy lies on the AA schedule of Figure 6,[9] at a point at which the

[8] Considerations of symmetry may suggest to the reader that two further patterns are possible: (i) ψ_1 (t) = 0 = ψ_2 (t) over a finite interval; (ii) ψ_2 (t) = 0 over a finite interval. In fact, however, both possibilities may be dismissed. From equations (28) and (29) it is easily shown that case (i) will only occur at the point (π^T, W^T). As for case (ii), it can be demonstrated after some rather tedious manipulations that the trajectories corresponding to it play no part in the optimal solution, and hence it is not discussed further in this paper.

[9] Since our concern in this paper is with the identification of the optimal deflationary strategy, we shall assume that the initial position (which is assumed to constitute a steady-state equilibrium corresponding to arbitrarily given feasible values of μ and d) lies above and to the right of (π^T, W^T).

solution for μ from equation (30) does not entail the violation of the bounds on this instrument, the optimal policy would simply involve a movement along this schedule to the target position. Next, we examine the possibility of 'switches' from other patterns into Pattern V when these conditions are not met. We shall first demonstrate that a switch from Patterns II or III to Pattern V (or vice versa) is impossible. In Pattern II, we know that ψ_1 and ψ_2 are both positive, while in Pattern V, $\psi_1 = 0$ and ψ_2 is (for $W > W^T$) strictly negative. Suppose that a switch-point from Pattern II to Pattern V exists: since ψ_1 and ψ_2 are both continuous functions of time, immediately prior to arriving at such a point each of these variables can be made arbitrarily close to zero. However, at the switch-point ψ_2 is required to be discretely negative: by continuity this is impossible, and hence such a switch-point does not exist. An analogous proof may be employed to demonstrate the impossibility of a switch between Patterns III and V.

On the other hand, switches between Patterns I, IV, and V are eminently feasible. Moreover, it is easily shown that, at common points, the $W - \pi$ trajectories of Pattern IV are steeper than the AA locus of Figure 6 (in the region in which the latter corresponds to feasible values of μ), which in turn has an algebraically greater slope than the $W - \pi$ trajectories of Pattern I. The set of possible scenarios may therefore be characterized by Figure 7.

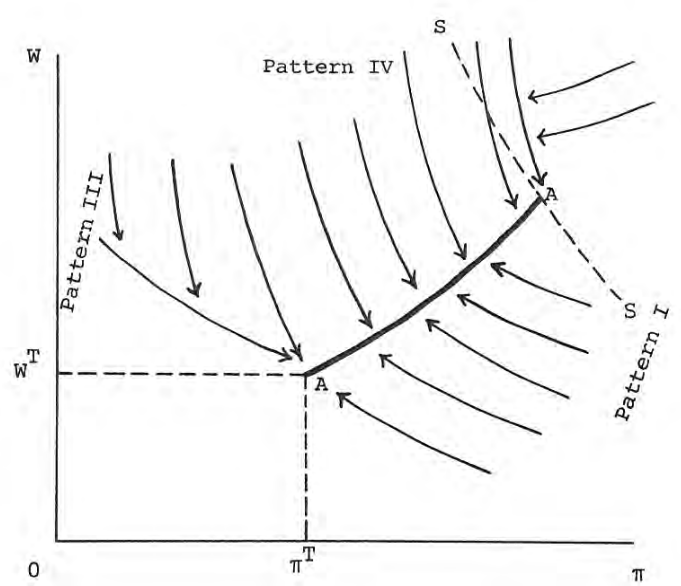

Figure 7

The AA schedule is identical to that of Figure 6 above, while the area to the north-east of the SS curve depicts the region (if it exists) in which the application of a Pattern V policy entails infeasible values of μ. Thus, only that portion of the AA schedule that lies below SS may be incorporated into an optimal strategy.

From Figure 7 it is evident that the complete optimal strategy partakes of one of several alternative forms, depending upon the location of the initial conditions. If the economy is initially at a point that lies to the right of AA, it is required first of all to follow a Pattern I policy, followed by a switch to a Pattern V policy (if the Pattern I locus meets AA below the SS curve), or to a Pattern IV policy (in the converse case), and thence to a Pattern V policy (the Pattern IV trajectory chosen would be that which meets AA precisely at its intersection with SS). In general, if the economy lies initially above SS and to the right of this last-mentioned Pattern IV trajectory, it will follow the same sequence, while if it lies above SS and to the left of this trajectory it will pursue a Pattern IV policy until it reaches the AA locus. Next, if initially the economy is at a point on AA and below SS it would follow a Pattern V policy throughout. Lastly, if the economy were initially to the left of AA and below SS, it would first employ a Pattern IV policy,[10] followed by a switch to either Pattern V, or (if the Pattern IV trajectory leads directly to (π^T, W^T)) to $\mu = \mu^T$ at the precise instant at which the target is attained, or to a Pattern III policy leading to the target, at which point a switch would be made to (μ^T, d^T). In Figure 7, we have depicted the Pattern III phase as being characterized by falling W and rising π: if, however, the target were to lie below the $\dot{W} = 0$ locus of Figure 4, then, during at least part of the Pattern III phase, the economy, having to 'overshoot' the target, would then be experiencing rising levels of W and π.[11,12]

[10] In Figure 7, we have depicted the Pattern IV policy as involving declines in both W and π. If, however, the policy were instituted at a point that lies between the $\dot{W} = 0$ and $d\pi/dt = 0$ loci of Figure 5 (and to the right of their intersection point), the economy would first experience a phase of rising W and falling π, followed by one of falling W and falling π.

[11] In this latter case, one might speculate that a switch might to made from Pattern IV to Pattern II instead. This, however, may be shown to be impossible as follows. During Pattern IV both ψ_1 and ψ_2 are negative, while during Pattern II they are both positive. Since these variables are continuous functions of time, it follows that at a switch point from Pattern IV to Pattern II they are each equal to zero: moreover, at such a point $\dot{\psi}_1$ must be positive, since $\psi_1 > 0$ in Pattern II. However, from the former of equations (19b), we see that $\dot{\psi}_1 < 0$ when $\psi_1 = 0 = \psi_2$ and $W < W^T$ (as it must be at any point below the target). This contradiction establishes the impossibility of such a switch. In fact, it appears that Pattern II plays no part in any deflationary strategy.

[12] We are now in a position to briefly outline the proof of the sufficiency of the Pontryagin

IV

Certain basic insights may be gleaned from the technical analysis of the preceding section as to the general features of an optimal deflationary strategy for less developed economies. These may, in conclusion, be summarized as follows:

(1) Except for the period, if it is at all called for, during which a Pattern III policy is applied, the optimal strategy entails that d be raised to and maintained at its upper bound throughout. (This of course implies that the actual level of d be gradually reduced as π falls). The critical importance of this instrument in the stabilization process is thus amply borne out by our analysis.

(2) While the maintenance of d at its upper bound is necessary if the economy is to receive an adequate deflationary stimulus, it is by itself insufficient to 'steer' the economy to the exact desired steady-state equilibrium. For this purpose, the second instrument, μ, must also be actively manipulated from the commencement of the program. If the rate of inflation prevailing in the original 'financially repressed' equilibrium is considerably higher than the target rate, then μ will, as in Pattern I, initially have to be reduced to its lower bound. On the other hand, if the required amount of deflation is quite moderate, then we obtain the surprising result that (as in the case of Pattern IV and the initial stages of Pattern V) μ has to be increased instead. Moreover, even the significant decreases in μ initially called for under Pattern I are reversed at a subsequent stage of the transition.

(3) The deflationary force of these policies manifests itself from the outset of the stabilization program. As Figure 7 clearly indicates, π either declines monotonically to its target value, or 'overshoots' it and then converges upward to it.[13]

necessary conditions in the context of the present model. From Arrow and Kurz (op. cit., Chapter II, Proposition 8), it is known that this will be the case if (a) the maximized Hamiltonian is a concave function of the state variables, and (b) the transversality conditions are satisfied. Condition (a) may be easily verified to hold in each pattern of the optimal strategy, bearing in mind the signs of ψ_1 and ψ_2 in each pattern. The satisfaction of condition (b) follows from the fact that both of these variables are equal to zero at the target point, since only then can μ assume the interior value μ^T.

[13] The behaviour of the other state variable W, and hence of the velocity of circulation of money V, exhibits somewhat greater variety. Ordinarily, one would expect V to decline during the deflationary process: however, during either the Pattern I or the Pattern IV phase, it is possible that V may rise for some time. The cause in each case is, however, different: during, Pattern I any rise in V would be attributable to the fact that μ is reduced discretely, while π can only fall continuously, while in

(4) This sustained process of price deflation occurs without any associated loss of real output. In the Pattern I phase, real output is constant, by definition of the lower bound on μ: however, in all other phases the economy is experiencing positive growth rates of real output. During the Pattern IV phase, in particular, the growth rate is likely to be quite high, since μ is set equal to its upper bound, while π and (at least after some time) W are both declining. A suitably orchestrated process of price deflation is, therefore, capable of ensuring a considerably more felicitous trade-off between the objectives of deflation and economic growth than has generally been appreciated.[14]

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Pattern IV a rise in V would be due to an increase in the growth rate γ (as discussed further below) to such an extent that increases in real output Y temporarily exceed increases in the real money supply M/P (note that M/P falls during the phase of increasing V in Pattern I). In either case, however, V must after some time commence declining, although as mentioned earlier there remains the possibility of over-shooting in the Pattern III phase.

[14] These conclusions are, it should be noted, substantially independent of the assumption that expectations are rational. The present author has, in a separate paper, investigated the optimal stabilization path in the adaptive expectations model summarized by equations (10) - (11) above. The resulting technical analysis is somewhat more complicated, since in this model an explicit distinction has to be drawn between π and π^* : however, the qualitative results are virtually identical to those reported above. One could, in similar vein, relax a number of other assumption of the model. For example, it is possible that q, and hence the money multiplier, could increase up to some asymptote should the government choose to relax its reserve requirements somewhat. One could also modify equation (5) by permitting a certain proportion of net investment out of business profits, as well as from bank loans. These changes would complicate the mathematical analysis, but would not change qualitative results. In fact, experimentation along these lines strongly suggests that our optimization analysis has uncovered policy guidelines which are broadly applicable to a wide range of developing economies.

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